DISCREPANCY THEORY

Reading Group, May 6.

INTRO:

$$V = \{1, \ldots, n\}$$

$$\mathcal{S} = \{S_1, \ldots, S_m\}, \quad S_i \subseteq [n], \ \forall i \in [m]$$

$$\chi : V \longmapsto \{-1, +1\}$$

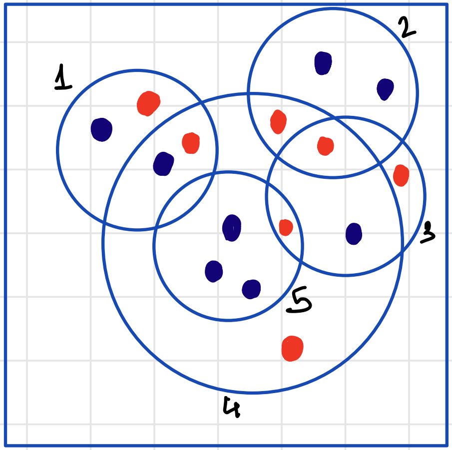
$$\chi(S_i) = \left| \sum_{j \in S_i} \chi(j) \right|$$

$$\mathrm{disc}(\mathcal{S}) = \min_{\chi} \max_{i \in [m]} \chi(S_i)$$

Eg:

$\chi(S_1) = 0$

$\chi(S_5) = 2$



RESULTS:

$\rightarrow$ Non Constructive.

* **Theorem** (Spencer '85): $|V| = n$, $|S| = m$

$\exists \chi : V \to \{-1, 1\}$ s.t. $\chi(S) < k\sqrt{n \log_2(m/n)}$.

$k = 5$ for $m = n$.

$\rightarrow$ Constructive.

* **Theorem** (Bansal '10): Randomized poly time algo. to find coloring with disc $O(\sqrt{n \log(m/n)})$.

SPENSER'S RESULT:

* <u>Theorem</u> (Spencer '85): $|V| = n$, $|\mathcal{S}| = m$

$\exists \chi : V \to \{-1, 1\}$ s.t. $\chi(\mathcal{S}) < K\sqrt{n \log_2(2m/n)}$.

<u>Spenser showed:</u> $\exists$ partial coloring $y \in \underline{\underline{\{-1, 0, 1\}^n}}$

s.t. $y(\mathcal{S}) \leq O(\sqrt{|V|})$, s.t. $\underline{|y| \geq n/2}$.

$\Rightarrow$ Suffices to prove the theorem:

1. Find a partial coloring. ($n/2$ elements are colored ± 1)

2. Recurse on remaining uncoloured elements.

$\Rightarrow$ <u>Step 1:</u> $n/2$ elements get colored with disc $O\left(\sqrt{n \log \frac{m}{n}}\right)$

<u>Step 2:</u> Remaining $\leq n/2$ elements get colored with disc $O\left(\sqrt{\frac{n}{2} \log \frac{2m}{n}}\right)$

$\vdots$

<u>Step $\log n$:</u> all elements get colored. with disc $O\left(\sqrt{\log m}\right)$

<u>Total Discrepancy:</u>

$$O\left(\sqrt{n \log_2 \frac{m}{n}}\right) + O\left(\sqrt{\frac{n}{2} \log \frac{2m}{n}}\right) + \dots + O\left(\sqrt{1 \cdot \log m}\right)$$

$$\approx \sqrt{n \log \frac{m}{n}}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{2^{\log n}}}\right) + c = O\left(\sqrt{n \log \frac{m}{n}}\right)$$

$\blacksquare$

$m = n$

$\therefore$ Partial Colouring Suffices.

RE-FRAMING PROBLEM:

$$V := [n], \qquad \mathcal{S} := \{S_1, \dots, S_m\}$$

$$A := \begin{array}{c} \\ S_1 - \\ S_2 - \\ \vdots \\ S_m \end{array} \begin{array}{c} \begin{array}{cccccccccc} \{1\} & \{2\} & \cdots & & & & & & & \{n\} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\ & & & & \vdots & & & & & \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad x := \begin{bmatrix} -1 \\ +1 \\ -1 \\ -1 \\ +1 \\ \vdots \\ -1 \end{bmatrix} \begin{array}{c} -\{1\} \\ -\{2\} \\ \vdots \\ \\ \\ \\ -\{n\} \end{array}$$

$$\operatorname{disc}(\mathcal{S}) = \min_{x \in \{-1,1\}^n} \|Ax\|_\infty .$$

$$x(S_i) = |A_i \cdot x|$$

VECTOR BALANCING:

$$A := \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & & & & & & v_n \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\ & & & \vdots & & & & & & \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad x := \begin{array}{c} \begin{bmatrix} -1 \\ +1 \\ -1 \\ -1 \\ +1 \\ \vdots \\ -1 \end{bmatrix} \end{array} \begin{array}{l} -\{1\} \\ -\{2\} \\ \vdots \\ \\ \\ \\ -\{n\} \end{array}$$

$$Ax = -1\begin{pmatrix} | \\ v_1 \\ | \end{pmatrix} + 1\begin{pmatrix} | \\ v_2 \\ | \end{pmatrix} - 1\begin{pmatrix} | \\ v_3 \\ | \end{pmatrix} - 1\begin{pmatrix} | \\ v_4 \\ | \end{pmatrix} \cdots - 1\begin{pmatrix} | \\ v_n \\ | \end{pmatrix}$$

$\therefore$ Given vectors $v_1, \ldots, v_n$, s.t. $\|v_i\| \leq O(\sqrt{m})$ find a signing $x \in \{-1,+1\}^n$ s.t.

$\|Ax\|_\infty$ is minimized.

CONVEX GEOMETRY VIEW (seemingly unrelated)

* **Theorem** (Giannopoulos '97)

Let $K \subseteq \mathbb{R}^n$ be a symmetric cvx. body s.t.

$\gamma_n(K) \geq e^{-\delta n}$ $\Rightarrow$ Gaussian measure of K, ie;

$$\left(\frac{1}{\sqrt{2\pi}} \int_K e^{-\|x\|^2/2} dx \geq e^{-\delta n} \right)$$

and $v_1, \ldots, v_m \in \mathbb{R}^n$ be $\|v_i\|_2 \leq \delta$.

Then $\exists$ partial coloring $y_1, \ldots, y_m \in \{-1, 0, +1\}$

$|\text{supp}(y)| \geq m/2$ s.t.

$$\sum_{i=1}^{m} y_i v_i \in 2K.$$

WHY, HOW CVX. GEOM. RELEVANT ???

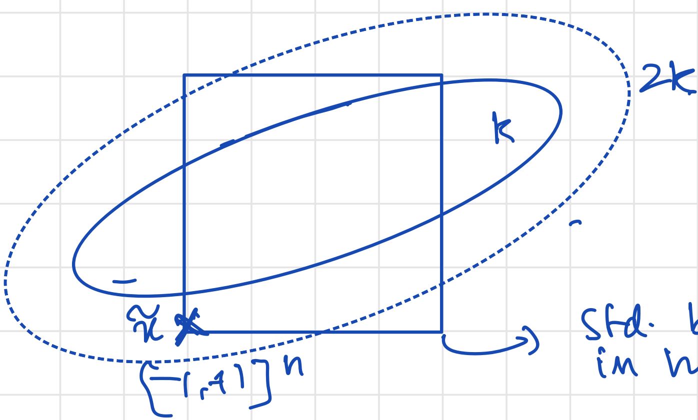
Consider $K = \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; \left|\sum_{j \in S_i} \underline{x}_j\right| \leq \underline{\Theta\left(\sqrt{n \log 2m/n}\right)} \;\; \underline{\forall i \in [m]} \right\}$

$x \in \{-1,1\}^n$

- $\underbrace{v_1, \ldots, v_n}_{} \in \mathbb{R}^n$ be standard basis vec.
- Let $\underline{y}$ be the partial coloring $\checkmark$ $(\mathrm{supp}(y) \geq n/2)$

$$\tilde{x} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ +1 \end{pmatrix} = \underbrace{\tilde{x} := -1\, v_1 + 1\, v_2 - 0\, v_3 - 1\, v_4 + 0\, v_5 \cdots + 1\, v_n}_{}$$

Giannopoulous $\Rightarrow$ $\tilde{x} \in 2K$ $\checkmark$ $\tilde{x} = \{\pm 1, 0\}$



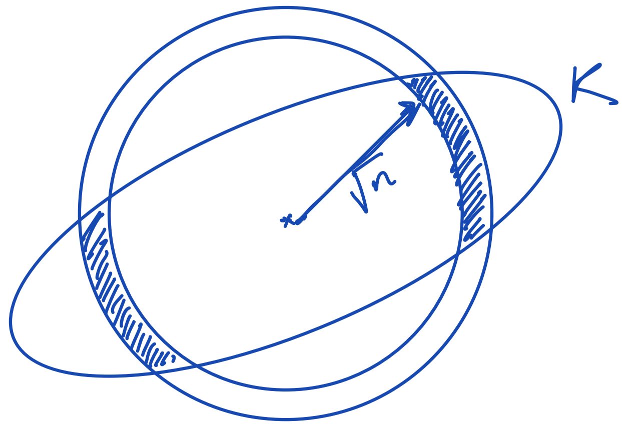
std. basis vectors in hypercube.

$\therefore$ We found a partial coloring $\tilde{x}$ s.t. $\underline{\tilde{x}(S)} \leq 2\,\underline{\Theta\left(\sqrt{n \log 2m/n}\right)}$

PROOF SKETCH: (Gia '97)

Let K s.t. $\gamma_n(K) \geq \frac{1}{2}$. (shaded region $\geq \frac{1}{2}$)

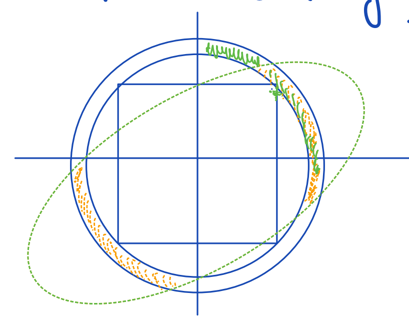
$\Rightarrow \mathbb{R}^n$



· Let $v_1, \dots, v_n$ be std. basis vectors (holds for any vec. actually)

· Consider all 2^n translates of K

i.e. $\sum_{i=1}^{n} \frac{y_i v_i}{4} + K$, $y_i \in \{\pm 1\}$



Total measure of translates: $(n \geq 7)$

$$\gamma_n\left(\sum_y \left(\sum \frac{y_i v_i}{4} + K\right)\right)$$

$$= \int_{\mathbb{R}^n} \sum_y \mathbb{1}\left\{\sum \frac{y_i v_i}{4} + K\right\}(x) \; \gamma_n(dx)$$

$$\geq \int_{\mathbb{R}^n} \sum_y \mathbb{1}\{K\} \; e^{-\frac{\|\sum y_i v_i\|^2}{16 \cdot 2}} \gamma_n(dx)$$

$$\geq 2^n e^{-n/32} \gamma_n(K) \qquad \left(\because \|\sum y_i v_i\|^2 = n\right)$$

$$\geq 2^{\left(1 - \frac{1}{32 \log 2} - \frac{1}{n}\right) n} \geq 2^{\underline{\underline{\left(1 - \frac{1}{32 \log 2} - \frac{1}{7}\right) n}}}$$

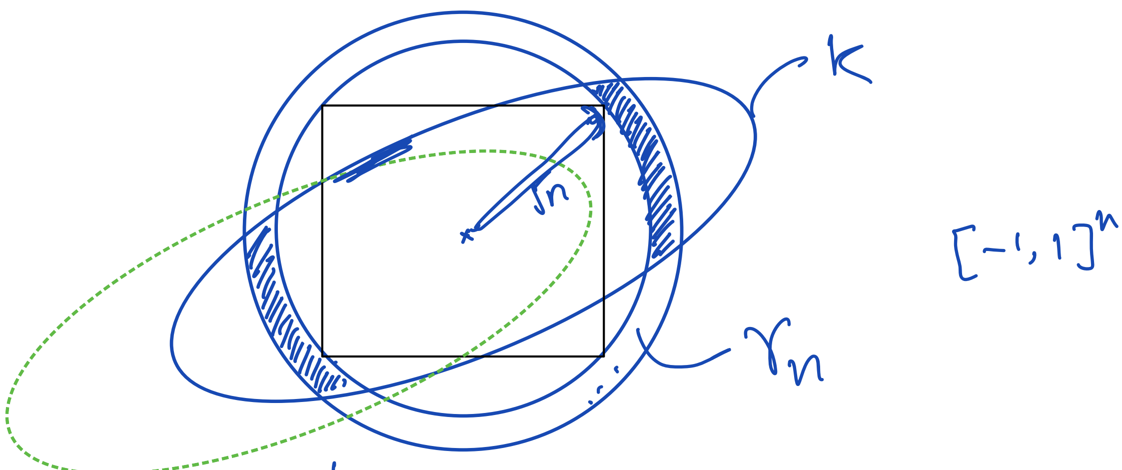
$$\geq \underline{\underline{2^{H(1/4) n}}}.$$

Where H is the binary entropy function

$$H(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha).$$

$\therefore$ The total measure of translates is HUGE!!!

$\Rightarrow$ There are a <u>LOT of overlaps</u> b/w translates.



$\sum y_i v_i + K$, for $y = (-1,-1,\dots,-1)$.

In fact, the number of overlaps are:

$\exists$ subset A of $\{-1,1\}^n$ with $|A| \geq 2^{H(1/4)n}$

s.t.

$$\bigcap_{y \in A} \left(\frac{\sum y_i r_i}{4} + K \right) \neq \emptyset \quad \text{(Pigeon-Hole)}$$

[Klietman's Result] $\Rightarrow$ $\exists$ $y', y'' \in A$ s.t.

$$\left|\{i : y'_i = y''_i\}\right| \leq n/2 \quad \& \quad \frac{\sum y'_i r_i - \sum y''_i r_i}{4} \in 2K$$

& Final partial coloring: $y = \frac{y' - y''}{2} \in 4K$. $\blacksquare$

* WHAT HAPPENS IF I COLOUR RANDOMLY?

- Recall: $A = \begin{pmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — a_m — \end{pmatrix}$, $a_i \in \mathbb{R}^n$. Find $x \in \{\pm 1\}^n$ s.t. $\|Ax\|_\infty$ is small.

- Let $x_1, \ldots, x_n \overset{iid}{\sim} \text{unif } \{\pm 1\}$.

<u>Chernoff:</u>

$\left(\mathbb{P}(|a_j^T x| \geq t) \leq 2e^{-t^2/2n} \right)$

$$\mathbb{P}\left(|a_j^T x| \geq \lambda_j \sqrt{n} \right) \leq 2e^{-\lambda_j^2/2}$$

<u>Union Bd:</u>

$$\mathbb{P}\left(|a_j^T x| \leq \lambda_j \sqrt{n}, \forall j \in [m] \right) \geq 1 - 2\left(e^{-\lambda_1^2/2} + \cdots + e^{-\lambda_m^2/2} \right)$$

$\therefore$ If $\displaystyle\sum_{j=1}^{m} e^{-\lambda_j^2/2} < \frac{1}{2} \Rightarrow \mathbb{P}\left(|a_j^T x| \leq \lambda_j \sqrt{n}, \forall j \right) > 0$

Since we want uniform bound

$$\lambda_j \geq \sqrt{2 \log 2m} \quad , \quad \forall j \in [m]$$

$\because$ Probabilistic method gives

$$\|Ax\|_\infty = \max_j \lambda_j \sqrt{n} = O\left(\sqrt{2n \log 2m}\right)$$

$\because$ Random coloring gives $O\left(\sqrt{n \log m}\right)$

Spencer gave $O\left(\sqrt{n \log 2m/n}\right)$

$\Rightarrow$ for $m = n$,

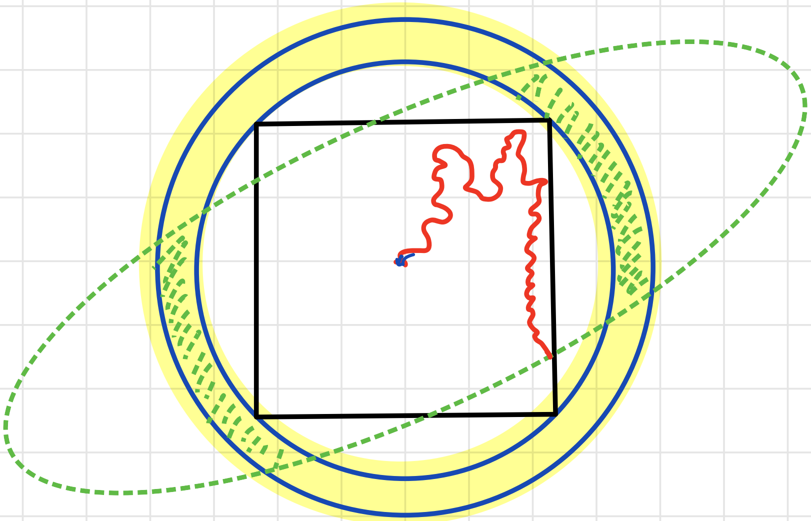
random coloring $= O\left(\sqrt{n \log n}\right)$

Spencer $= O\left(\sqrt{n}\right)$.

* CONSTRUCTIVE DISCREPANCY MINIMIZATION

(Lovette Meka)

Idea: Recall the figure:



$\because$ Cvx body has large measure (we'll see)

$\Rightarrow$ hypercube $[-1,1]^n$ is mostly inside body.

$\Rightarrow$ Random point on surface of $[-1,1]^n \cap K$ will be many ± 1 coordinates.

$\approx$ fractional partial coloring.

$\therefore$ Algo: Start random walk from center till you hit face then do r.w. along that face till you stop.

FORMALLY:

* **Theorem (Lovett-Meka '12):** Let $a_1, \ldots, a_m \in \mathbb{R}^n$, $x_0 \in [-1,1]^n$ be "starting point". Let $\lambda_1, \ldots, \lambda_m \geq 0$ be thresholds s.t. $\sum_{j=1}^{m} \exp(-\lambda_j^2/16) \leq \frac{n}{16}$. Then

$\exists$ rand. algo w.p. 0.1 finds $x \in [-1,1]^n$ s.t.

(i) $|\langle x - x_0, a_j \rangle| \leq \lambda_j \|a_j\|_2$

(ii) $|x_i| \geq 1 - \delta$, for at least $n/2$ indices $i \in [n]$, for some small $\delta > 0$.

DIGESTING THE THEOREM

Let $\|a_1\| = \cdots = \|a_m\| = 1$

$$P := \left\{ x \in \mathbb{R}^n : \underbrace{|x_i| \leq 1 \;\; \forall i \in [n]}_{\text{Variable constraint}}, \; \underbrace{|\langle x - x_0, a_j \rangle| \leq \lambda_j}_{\text{Discrepancy Constraint}} , \; \forall j \in [m] \right\}$$

The theorem can be rephrased as:

$\exists \; x \in P$ which satisfy $n/2$ variable constraint tightly (without any slack)

Algo: random walk till you hit a constraint. Then walk in orthogonal subsp.

$\Rightarrow$ As long as $\sum \exp(-\lambda_j^2) \ll n$, the r.w. hits many variable constraint.

<u>ALGO:</u>

$$C_t^{var} := C_t^{var}(X_{t-1}) = \{ i \in [n] : (X_{t-1})_i \geq 1 - \delta \}$$

$$C_t^{disc} := C_t^{disc}(X_{t-1}) = \{ i \in [n] : \langle X_{t-1} - x_0, a_j \rangle \geq \lambda_j - \delta \}$$

$$\mathcal{V}_t := \mathcal{V}_t(X_{t-1}) = \text{orthogonal subsp. to } C_t^{var}, C_t^{disc}$$

$$= \left\{ u \in \mathbb{R}^n : \begin{array}{ll} u_i = 0 & \forall i \in C_t^{var} \\ \langle u, a_j \rangle = 0 & \forall j \in C_t^{disc} \end{array} \right\}$$

$$X_t = X_{t-1} + \gamma U_t .$$

Run the procedure for $X_1, \dots, X_T$. $T = O(1/\gamma^2)$.

- Parameter Settings:

$$\gamma \leq \frac{\delta}{\sqrt{C \log(mn/\delta)}}$$

$$T = O\left(\frac{1}{\gamma^2}\right)$$

$\gamma \Rightarrow$ small step size. $\Big\}$ affects running time of algo.

$\delta > 0$ is also small.

PROOF SKETCH

1. w.h.p. $x_1, \ldots, x_T \in P$

→ If it violates this condition at any step, ⇒

$$x_t = x_{t-1} + r_t \underbrace{u_t}_{}$$

↪ large ⇒ low prob.

PROOF SKETCH

2. Since $\sum \exp\left(-\lambda_j^2/16\right) \leq n/16$

$$\mathbb{E}\,|C_T^{disc}| \ll n.$$

Because: $\mathbb{E}\,|C_T^{var}| = \Omega(n)$

If $(t<T)$, $|C_t^{var}|$ is big we are done.

if $|C_t^{disc}|$ is small $\Rightarrow \dim(V_{t-1})$ large

$\Rightarrow \mathbb{E}\,\|X_t\|^2$ increases significantly

& $\|X_t\|^2 \leq n \quad (\because X_1, \dots, X_T \in P)$

$\Rightarrow \mathbb{E}\,|C_T^{var}|$ will be large.

* SOME INTUITION ABOUT $\sum \exp(-\lambda_j^2) << n.$

$$K = \{ x \in \mathbb{R}^n \mid |\langle x, a_j \rangle| \le \lambda_j \quad \forall j \in [m] \}$$

$$S_i = \{ x \in \mathbb{R}^n \mid |\langle x, a_i \rangle| \le \lambda_i \}, \quad \forall i \in [m]$$

$$K = \bigcap_{i \in [m]} S_i$$

$$\gamma_n(K) \geq \prod_{i=1}^{m} \gamma_n(S_i) \geq \prod_{i=1}^{m} \exp\left(-2 e^{-\lambda_i^2/2}\right)$$

$$\geq e^{-n/500}.$$

$$\gamma_n(S_i) = \Phi(\lambda_i) \geq 1 - e^{-\lambda_i^2/2} \geq \exp\left(-2 e^{-\lambda_i^2/2}\right)$$

In fact: $\sum \exp(-\lambda_i^2/16) \leq \frac{n}{16}$,

- I can set constant number of λ_i to be 0, ie 0 disc for $\Omega(n)$ sets.
- Compare with random coloring:

  We needed $$\sum \exp(-\lambda_i^2/2) < \frac{1}{2}.$$

$\Rightarrow$ Much stronger than random coloring.