Intro to Free Probability Theory

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My Motivation

- Given two matrices : A & B (with eigenvals/vecs)
- What can we say about the eigenvalues of

$$\underline{A+B} \text{ (or) } A \cdot B$$

- Depends on the eigenvalues & eigenvectors of A & B
- Tool(s) I know :
  1. Trace method.
  2. Weyl's ineq.

  $\vdots$

  } not always useful.

$$\mathbb{E}[(A+B)^2] = \mathbb{E}[A^2] + \mathbb{E}[B^2] + 2\,\mathbb{E}[A]\,\mathbb{E}[B]$$

What if Eigenvectors are "uncorrelated".

Look at $A + Q^T B Q$ — Q is a random rotation matrix

eigenvecs of B: $b_1, \dots, b_n$

" " $Q^T B Q$: $Q^T b_1, \dots, Q^T b_n$

Can we say something about eigenvals of $A + Q^T B Q$ now?

"Takes eigenvecs out of the question".

→ can reason about spectrum based on eig(A) & eig(B).

Free Probability Result:

As size of matrix $\to \infty$, we have

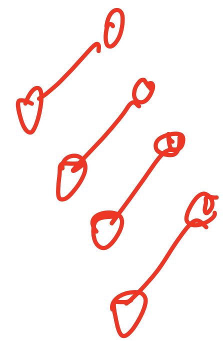
$$\text{eigenvals}\left(A + \underline{\underline{Q}}^T B Q\right) = \text{Unif}(\text{eig}(A)) \boxplus \text{Unif}(\text{eig}(B))$$

free additive convolution

Why Care About $A + Q^T B Q$?

- Let P be a random permutation matrix.
  Let's look at $A + P^T B P$

- Let M be a perfect matching matrix:



$$A_G := P_1^T M P_1 + P_2^T M P_2 + P_3^T M P_3$$

this creates a 3-regular graph
(maybe multi-edges, but prob. decreases as size of graph increases)

[MSS] : you can bound any one eigenvalue of A_G "nicely".

[MSS] $\rightarrow$ $\mathop{\mathbb{E}}_{Q} \chi_x(A + Q^T B Q) = \mathop{\mathbb{E}}_{P} \chi_x(A + P^T B P)$

[MSS] : $\exists$ "good" P s.t.

$$k\text{-th root}\left(\mathop{\mathbb{E}}_{Q} \chi_x(A + Q^T B Q)\right) \leq k\text{-th root}\left(\chi_x(A + P^T B P)\right).$$

$$\max\left(\chi_x(A + P^T B P)\right) \leq \max\left(\mathop{\mathbb{E}}_{Q} \chi_x(A + Q^T B Q)\right)$$

Free Probability Basics

$\exists$ one more prob. theory .

(We work with moments of distributions)

- Freeness $\simeq$ Independence
- Central limit theorem
- Some analytic tools

Classical Prob. Thy.

Classical independence gives a way of determining mixed moments by marginals.

eg: $\mathbb{E}[abab] = \mathbb{E}[a^2]\,\mathbb{E}[b^2]$ $\quad$ a & b are indep.

Freeness

Def: [Non-commutative Prob space [ncps]].

$$(\mathcal{A}, \varphi)$$

- $\mathcal{A}$: unital $\mathbb{C}$-algebra (ring & $\mathbb{C}$-vector space) $\quad \underline{1 \in \mathcal{A}}$ $\quad \underline{\varphi(1) = 1}$

  $\mathcal{A}$ has $a, b, a^{-1}, a+b, \ldots$

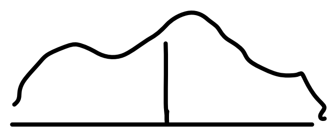
- $\underline{\underline{\varphi : \mathcal{A} \to \mathbb{C}}}$

- φ is tracial : $\varphi(ab) = \varphi(ba)$

- Elements $a \in \mathcal{A}$ are called (nc) random var.

- "Think of $\varphi(a) = \mathbb{E}[a]$ "

Eg: $\mathcal{A}$: $n \times n$ matrices
φ : normalized trace



$$\underline{\varphi(a) = \frac{1}{n} \sum_{i=1}^{n} a_{ii}} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i(a) \longrightarrow \underset{\lambda \sim \text{spec}(a)}{\mathbb{E}[\lambda]}$$

$$\varphi(I) = 1$$

NCPS: Also equipped with * operation

$$*: \mathcal{A} \to \mathcal{A}$$
$$a \mapsto a^* \qquad \text{s.t.}$$

- $(a^*)^* = a$
- $(ab)^* = b^* a^*$
- $(a+b)^* = a^* + b^*$
- $\varphi(a^* a) \geq 0$ with equality iff $a = 0$.

* With this, we can define a to be :

- Self adjoint : $a = a^*$
- Unitary : $a^* a = a a^* = 1$
- Normal : $a a^* = a^* a$

eg: We can prove we get inprod $\therefore$ prove things like C-S ineq

$$|\varphi(b^* a)|^2 \leq \varphi(a^* a)\, \varphi(b^* b).$$

Recall: · φ encodes moments.
· Algebra encodes relation b/w r.v.s

Analytic Dist:

Let (A, φ) be a ncps, $a \in \mathcal{A}$ normal. If $\exists$ cpt. supp dist μ on $\mathbb{C}$ s.t.

$$\int z^k \bar{z}^l \, d\mu(z) = \varphi\left(a^k (a^*)^l\right) \quad \forall\, k, l \in \mathbb{N},$$

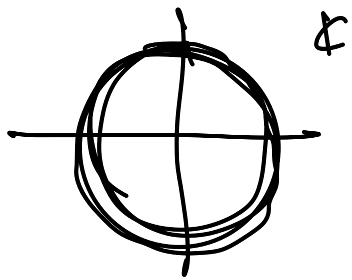
then μ is uniquely determined & is called analytic dist. of a.

Eg / Def : Haar Unitary :

Let (A, φ) be a ncps. $u \in A$ is Haar unitary if it is unitary and

$$\boxed{\varphi(u^k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{o.w.} \end{cases}}$$

The corresponding analytic dist. is the unif dist over unit circle $\mathbb{C}$. (verify)



Def: Let $(\mathcal{A}, \varphi)$ be a ncps. $a, b \in \mathcal{A}$ are free if all centered alternating mixed moments vanish.

Eg: $\varphi(\bar{a}\,\bar{b}) = 0$ $\qquad\qquad a - \varphi(a) = \bar{a}$

$\varphi(\overline{a^2}\,\overline{b^n}\,\bar{a}\,\overline{b^3}) = 0$

Eg: 1

$$0 = \varphi((a - \varphi(a))(b - \varphi(b)))$$

$$= \varphi(ab - \varphi(a)b - a\varphi(b) + \varphi(a)\varphi(b))$$

$$= \varphi(ab) - \varphi(a)\varphi(b) - \varphi(a)\varphi(b) + \varphi(a)\varphi(b)$$

$$\Rightarrow \varphi(ab) = \varphi(a)\varphi(b)$$ $\Rightarrow$ Like in classical case.

2: $\underline{\varphi(\bar{a}\bar{b}\bar{a}\bar{b}) = 0}$ $\Rightarrow$

$$\varphi\big((a-\varphi(a))(b-\varphi(b))(a-\varphi(a))(b-\varphi(b))\big) \cdots$$

We get:

$$\underline{\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2.}$$

<u>In classical indep:</u>

$$\underline{\varphi(abab) = \varphi(a^2)\varphi(b^2)}.$$

<u>Why haven't we seen freeness:</u>

- Commuting r.v. are free only if one of them is a constant.
- Freeness is an ∞-dimn. phenomenon.

(Random matrices are "asymptotically" free.)

Eg: Let $(\mathcal{A}, \varphi)$ be ncps. Let $a, b, u \in \mathcal{A}$ s.t.

u is Haar unitary that is free from $\{a, b\}$

(Think as random rotation)

Then a and $u^* b u$ are free.

Central Limit Theorem (both for independence & free r.v.)

CLT asks about:

$$\lim_{N\to\infty} \frac{a_1+\cdots+a_N}{\sqrt{N}} \to \begin{array}{l}\varphi(a_i)=0\\ \varphi(a_i^2)=1\end{array}$$

all a_i have same distribution

where convergence is in terms of moments (weak cvg.):

$a_N \to a$ means $\forall m$ $\lim_{N\to\infty} \varphi_N(a_N^m) = \varphi(a^m)$.

- We want to focus on the moments.

Fix $m, N \geq 1$, m is finite, N will go to infinity later

Look at:

- $\varphi((a_1+\cdots+a_N)^m)$

Analyzing $\varphi((a_1+\cdots+a_N)^m)$

- $\varphi((a_1+\cdots+a_N)^m) = \sum_{\gamma:[m]\to[N]} \varphi(a_{\gamma(1)}\cdots a_{\gamma(m)})$

$(a_1+a_2)(a_1+a_2)(a_1+a_2)$

But note that:

$$\varphi(a_1 a_2 a_2 a_3 a_1 a_2) = \varphi(a_9 a_1 a_1 a_5 a_9 a_1)$$

We encode this information as follows:

Let $\pi = \{V_1, \dots, V_m\}$ be a partition of $[m]$

where

$\forall i,j \quad \gamma(i) = \gamma(j) \iff \exists l \quad i,j \in V_l$

Eg:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ a_1 & a_2 & a_2 & a_3 & a_1 & a_2 \\ a_9 & a_1 & a_1 & a_5 & a_9 & a_1 \end{array}$$

corresponds to $\underbrace{\{\{1,5\},\{2,3,6\},\{4\}\}}$

$$\underline{\binom{N}{|\pi|}\, \pi!}$$

- $\varphi((a_1+\cdots+a_N)^m) = \sum_{r:[m]\to[N]} \varphi(a_{r(1)}\cdots a_{r(m)})$

with this, we get:

$$\varphi((a_1+\cdots+a_N)^m) = \sum_{\pi:\text{ partition of }[m]} K_\pi \, A_\pi^{(N)}$$

$A_\pi^{(N)}$ → number of r's associated with π

→ only this depends on N

K_π ← common value of $\varphi(a_{r(1)}\cdots a_{r(m)})$ $\forall r$ associated with π

<u>Obvs</u>: π containing singletons does not contribute:

$$K_\pi = \varphi(\sim a_r \sim) = \underbrace{\varphi(a_r)}_{=0}\varphi(\sim\sim)$$

(both in classical & free prob.)

$\therefore$ We can restrict to π's with $|\pi| \le m/2$.

→ reason why $\sqrt{N}$ appears

Calculating $A_\pi^{(N)}$:

$A_\pi^{(N)}$: how many r's are associated with a π.

$$= N(N-1)\cdots(N-|\pi|+1)$$

because: let $\pi = \{\{abc\}, \{de\}, \{f\}\}$
partition corresponds to an idx. lets say j appearing at pos. a, b, & c.

$$\xrightarrow{N\to\infty} N^{|\pi|}$$

$$\Rightarrow \lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^m\right) = \lim_{N\to\infty} \sum_\pi K_\pi \frac{A_\pi^{(N)}}{N^{m/2}}$$

$$= \sum_\pi K_\pi \lim_{N\to\infty} \frac{A_\pi^{(N)}}{N^{m/2}}$$

$$= \sum_\pi K_\pi \lim_{N\to\infty} N^{|\pi|-m/2}.$$

(If $|\pi| < \frac{m}{2}$ this goes to 0)

$$\therefore \lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^m\right) = \sum_\pi \kappa_\pi \lim_{N\to\infty} N^{|\pi|-m/2} \qquad — (*)$$

- We had $|\pi| \leq m/2$ because singletons correspond to 0.
- If $|\pi| < m/2$, then $(*)$ goes to 0.

$\therefore$ Only pairings survive.

$$\therefore \lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^m\right) = \sum_{\substack{\pi:\text{ pairings}\\ \text{of } [m]}} \kappa_\pi .$$

Classical CLT

$\forall\, \pi$ pairings $\quad \kappa_\pi = \varphi(a_i^2) \cdot \varphi(a_j^2) \cdots \;= 1$

$\#$ pairings of $[m] := (m-1) \cdot (m-3) \cdots 1.$

- pick any elem. $(m-1)$ choices for its partner
- choose smallest elem in remaining & pair with any $(m-3)$ of them.

Then,

$$\frac{1}{\sqrt{2\pi}} \int t^m e^{-t^2/2}\, dt = \begin{cases} (m-1)(m-3)\cdots & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

$\blacksquare$

FREE CLT

$$\therefore \lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^m\right) = \sum_{\pi : \text{ pairings of } [m]} \kappa_\pi .$$

Play with freeness:

$\varphi(aabb) = \varphi(a^2 b^2)$

$\varphi(ab) = \varphi(a)\varphi(b)$

- $\pi = \{\{1,2\},\{3,4\}\}$

  1 2 3 4

  $\varphi(a^2 b^2) = \underline{\varphi(a^2)\varphi(b^2)}$

- $\pi = \{\{1,3\},\{2,4\}\}$

  1 2 3 4

  $\varphi(abab) = 0$

- $\pi = \{\{1,4\},\{2,3\}\}$

  1 2 3 4

  $\varphi(abba) = \varphi(a^2)\varphi(b^2)$

$$\varphi(ab^2a) = \varphi(a^2b^2) = \varphi(a^2)\varphi(b^2)$$

- Def: A pairing π is <u>non crossing</u> if $\forall \{p_1, p_2\}, \{q_1, q_2\} \in \pi$

$p_1 < q_1 < p_2 < q_2$ → doesnt happen

<u>Fact</u>: # of such pairings of $[2m]$ is C_m : the m-th Catalan number

<u>Fact</u>: The analytic dist. whose (even) moments are Catalan numbers is the <u>semicircular</u> dist.

$$\frac{1}{2\pi}\sqrt{4-x^2}$$