The Kadison-Singer Problem

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Reading Group Presentation: June 5, 2023.

KADISON-SINGER PROBLEM

Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on ℓ_2 have a unique extension to a pure state on $B(\ell_2)$, the von Neumann algebra of all bounded operators on ℓ_2?

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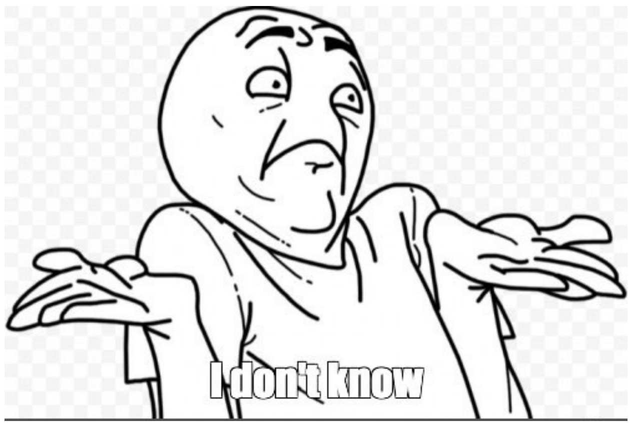


EQUIVALENTLY, . . .

* **Thm:** If $\varepsilon > 0$ and $v_1, \ldots, v_m$ are independent random vectors in $\mathbb{C}^n$ with finite support s.t.

$$\mathbb{E} \sum_i v_i v_i^* = I_n, \qquad \text{and}$$

$$\mathbb{E} \|v_i\|^2 \leq \varepsilon, \quad \forall i, \quad \text{then}$$

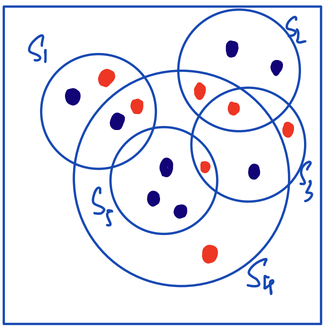
$$\mathbb{P}\left[\left\| \sum_{i=1}^m \hat{v}_i \hat{v}_i^* \right\| \leq \underline{(1+\sqrt{\varepsilon})^2} \right] \underline{> 0} .$$

- **Rmk:** Concentration Ineq $\Rightarrow$ $\underline{\| \sum v_i v_i^* \| \leq C(\varepsilon) \cdot \log(n)}$ whp.

DISCREPANCY THEORY

- **(SPENCER)**: Given sets $S_1, \dots, S_n \subset [n]$, colour elements of $[n]$ Red & Blue s.t.

$$\forall S_i : \quad \big|\, |S_i \cap R| - |S_i \cap B| \,\big| \leq 6\sqrt{n}$$



UNIFORMLY PARTITIONING VECTORS

Thm: Given vectors $v_1, \dots, v_m \in \mathbb{R}^n$, satisfying

$$\|v_i\|^2 \le \alpha, \quad \text{and}$$

$$\sum_{i=1}^{m} \langle v_i, x \rangle^2 = 1, \quad \forall \|x\| = 1$$

$\sum v_i v_i^T = I$

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there exists partition $T_1 \cup T_2 = [m]$:

$$\left| \sum_{i \in T_j} \langle v_i, x \rangle^2 - \frac{1}{2} \right| \le \underbrace{5\sqrt{\alpha}}, \quad \forall \|x\| = 1$$

↪ Discrepancy.

- non trivial guarantee only if $5\sqrt{\alpha} < 1/2$

UNIFORMLY PARTITIONING VECTORS

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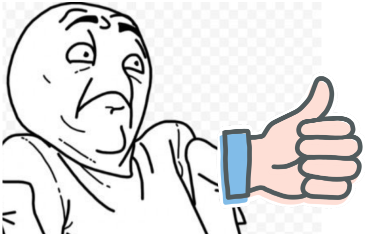
$\Downarrow$

Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on ℓ_2 have a unique extension to a pure state on $B(\ell_2)$, the von Neumann algebra of all bounded operators on ℓ_2?



INTERPRETING: Uniformly Partitioning Vectors

- Norm Bounds $\|v_i\|^2 \le \alpha$ is necessary:

  - Suppose $v_1, \ldots, v_m$ s.t. $\sum v_i v_i^T = I$, but $\|v_1\|^2 = 3/4$; $\forall\ \|v_i\|^2 \le \alpha$.

$\Rightarrow$ Partition T_1 contains v_1 $\Rightarrow$

$$\sum_{i \in T_1} \langle v_i, x \rangle^2 \ge \|v_1\|^2 = 3/4 .$$

$\therefore$ This partition has discrepancy at least $1/4$.

$\Rightarrow$ No way to get closer to $\frac{1}{2}$ when splitting v_1.

UNIFORMLY PARTITIONING VECTORS

Thm: Given vectors $v_1, \dots, v_m \in \mathbb{R}^n$, satisfying

$$\|v_i\|^2 \le \alpha, \quad \text{and}$$

$$\sum_{i=1}^{m} \langle v_i, x\rangle^2 = 1, \quad \forall \|x\| = 1$$

there exists partition $T_1 \cup T_2 = [m]$:

$$\left| \sum_{i \in T_j} \langle v_i, x\rangle^2 - \frac{1}{2} \right| \le 5\sqrt{\alpha}, \quad \forall \|x\| = 1$$

- Theorem says that the ONLY obstacle to obtaining low disc. solution is large vectors.

- **Rmk**: Disc. $O(\sqrt{\alpha})$ is tight.

$$\sum_{i=1}^{m} v_i v_i^T = I \quad \Rightarrow \text{ISOTROPY CONDITION}$$

(A normalization)

- $w_1, \ldots, w_m \in \mathbb{R}^n$ not isotropic.

s.t. $\text{Span}(w_1, \ldots, w_m) = \mathbb{R}^n$.

$$\Rightarrow \quad W = \sum_{i=1}^{m} w_i w_i^T \quad \Rightarrow \text{invertible}$$

$$\forall \; v_i = W^{-1/2} w_i$$

$$\Downarrow$$

$$\sum v_i v_i^T = W^{-1/2} \left(\sum w_i w_i^T \right) W^{-1/2} = I.$$

$$\forall \quad \|v_i\|^2 = \|W^{-1/2} w_i\|^2.$$

$$\|v_i\|^2 = \|W^{-1/2} w_i\|^2 \Rightarrow \text{Interpretation}$$

$$\|v_i\|^2 = \|W^{-1/2} w_i\|^2 = \sup_{x \neq 0} \frac{\langle x, W^{-1/2} w_i \rangle^2}{x^T x}$$

$$= \sup_{y = W^{1/2} x \neq 0} \frac{\langle W^{-1/2} y, W^{-1/2} w_i \rangle}{y^T W y}$$

$$= \sup_{y \neq 0} \frac{\langle y, w_i \rangle^2}{\sum_i \langle y, w_i \rangle^2} .$$

$\therefore$ $\|v_i\|^2$ measures max fraction of quadratic form of W that a single vector w_i can be responsible for.

* **Thm:** Given vectors $v_1, \dots, v_m \in \mathbb{R}^n$, satisfying

$$\|v_i\|^2 \leq \alpha \,, \quad \text{and}$$

$$\sum_{i=1}^{m} \langle v_i, x\rangle^2 = 1 \,, \quad \forall \|x\| = 1$$

there exists partition $T_1 \cup T_2 = [m]$:

$$\left| \sum_{i \in T_j} \langle v_i, x\rangle^2 - \frac{1}{2} \right| \leq 5\sqrt{\alpha} \,, \quad \forall \|x\| = 1 .$$

Idea 1: Randomly partitioning the vectors.
+
Matrix-Chernoff.

$\Rightarrow O(\sqrt{\alpha \log n})$ disc. whp

REMOVING log FACTOR

To remove the log factor, we use the following theorem:

$\{\tfrac{1}{2} v_1, \tfrac{1}{2} 0\}$

* <u>Thm</u>: If $\alpha > 0$ and $v_1, \dots, v_m$ are independent random vectors in $\mathbb{R}^n$ with finite support s.t.

$$\sum_{i=1}^{m} \mathbb{E}\, \hat{v}_i \hat{v}_i^{*} = \frac{I_n}{2}, \qquad \text{and}$$

$$\mathbb{E}\, \|\hat{v}_i\|^2 \le \alpha, \quad \forall i, \quad \text{then}$$

$$\mathbb{P}\left[\left\| \sum_{i=1}^{m} \hat{v}_i \hat{v}_i^{*} \right\| \le (1+\sqrt{\alpha})^2 \right] > 0$$

PROOF SKETCH

Theorem says: $\exists$ pt. $w \in \Omega$ (prob. sp.)
s.t.

$$\Big\| \sum_{i \le m} \hat{v}_i(w)\, \hat{v}_i(w)^T \Big\| \le (1+\sqrt{\alpha})^2$$

For every $w \in \Omega$, consider polynomial

$$P[w](x) := \det\Big(xI - \sum_{i \le m} \hat{v}_i(w)\, \hat{v}_i(w)^T \Big).$$

Note: $\sum \hat{v}_i \hat{v}_i^T$ is a symmetric

- $\| \sum \hat{v}_i \hat{v}_i^T \|$ is the largest root of the characteristic polynomial.
- characteristic polynomial has real roots.

PROOF SKETCH

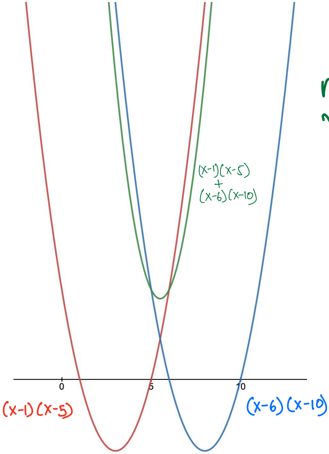
$$P[w](x) := \det\left(xI - \sum_{i \le m} v_i(w) v_i(w)^T \right)$$

STEP 1: $\exists\, w \in \Omega$ s.t.

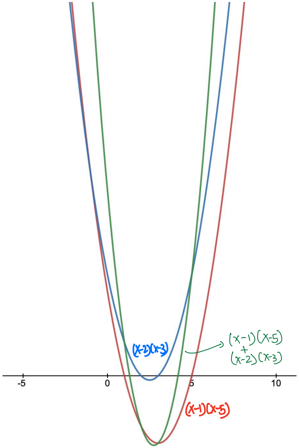
$$\lambda_{max}(P[w]) \le \lambda_{max}(\mathbb{E} P) \rightarrow \text{probabilistic method vibes}$$

· Roots of sum of polynomials don't have much to do with roots of individual polynomials.

Eg:



no real roots



real roots

STEP 2 : Upper bound roots of expected polynomials

$$\mu(x) := \mathbb{E} P(x)$$

- $\mu(x)$ = linear transform of m-variate polynomial $Q(z_1, \dots, z_m)$

  - Q does not have any roots in certain region of $\mathbb{R}^m$ $\rightarrow$ and use barrier functions.

$$\Downarrow$$

uses theory of "real stable" polynomial.

- We'll focus on STEP 1 & show it is sufficient to bound roots of Expected characteristic polynomial.

INTERLACING

f: degree n polynomial with real roots $\{\alpha_i\}$.

g: degree n or $n-1$ with all real roots $\{\beta_i\}$

g interlaces f if their roots alternate

$$\beta_n \le \alpha_n \le \beta_{n-1} \le \cdots \le \beta_1 \le \alpha_1 .$$

NOTATION: $g \longrightarrow f \Rightarrow$ largest root belongs to f.

- If a single g interlaces a family $f_1, \dots, f_m$

  $\Rightarrow$ Have a common interlacing.

• $f_1, \dots, f_n$ has common interlacing $\Leftrightarrow$ Every pair has common interlacing

$\Updownarrow$

$I_n \le I_{n-1} \le \dots \le I_n$ closed intervals

i-th root of f_j is contained in I_i.

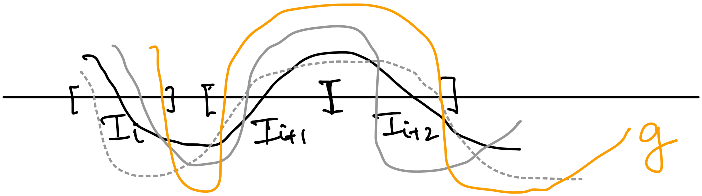
* Thm: Suppose $f_1, \dots, f_m$ are real-rooted of degree n with positive leading coef. $\lambda_k(f_j)$: k-th largest root of f_j

μ be any dist on $[m]$.

If $f_1, \dots, f_m$ have common interlacing

Then $\forall k = 1, \dots, n$

$$\min_j \lambda_k(f_j) \le \lambda_k\Big(\mathop{\mathbb{E}}_{j\sim\mu} f_j\Big) \le \max_j \lambda_k(f_j).$$



Proof: · At some point, it is all positive in I_i

· At some point, it is all negative in I_i

$\therefore$ This bound holds $\blacksquare$

Interlacing helps us achieve step 1:

STEP 1: $\exists\, \omega \in \Omega$ s.t.

$P[\omega](x) := \det\left(xI - \sum_{i \le m} v_i(\omega) v_i(\omega)^T\right)$

$\lambda_{max}(P[\omega]) \le \lambda_{max}(\mathbb{E} P)$ $\to$ probabilistic method vibes

FINDING COMMON INTERLACER

(Annoying)

Translate "interlacing" to "real-rootedness".

* **Thm:** Let $\{f_i\}$ be degree n monic polynomials.

The following are equivalent:

$\boxed{A.}$ All convex combinations $\sum \mu_i f_i$ has d-real roots.

$\boxed{B.}$ The collection $\{f_i\}$ has common interlacer.

The condition $\boxed{A.}$ is easy to work with.

Pf: Common interlacing is pairwise phenomenon.
(can skip)
consider polynomials f_0 & f_1

$$f_t := (1-t) f_0 + t f_1 \quad , \quad t \in [0,1]$$

- f_0 & f_1 ⇒ no common roots, wlog.
- t varies, roots of f_t : n continuous curves in cplx plane $c_1, \dots, c_n$ beginning at root of f_0 & ending at root(f_1)
- Curves must lie in real line, and no curve can cross roots of f_0 or f_1 in middle.

  i.e, $f_t(r) = 0$ & $f_0(r) = 0$ ⇒ $f_1(r) = 0$ ⇒⇐
- ∴ Each curve is an interval (non-overlapping)

  ⇒ Common interlacing.

RECALL THE THEOREM

* <u>Thm</u>: If $\alpha > 0$ and $\hat{v}_1, \ldots, \hat{v}_m$ are independent random vectors in $\mathbb{R}^n$ with finite support s.t.

$$\sum_{i=1}^{m} \mathbb{E}\, \hat{v}_i \hat{v}_i^* = I_n, \qquad \text{and}$$

$$\mathbb{E}\, \|\hat{v}_i\|^2 \le \alpha, \quad \forall i, \quad \text{then}$$

$$\mathbb{P}\left[\ \left\| \sum_{i=1}^{m} \hat{v}_i \hat{v}_i^* \right\| \le (1+\sqrt{\alpha})^2 \right] > 0. \qquad (*)$$

- Each v_i is a random vector, and we need to show that there exists non-0 prob. of $(*)$ happening.

Let

$$\theta(\hat{v}_1, \ldots, \hat{v}_n) = \min_{v_i \in \mathrm{supp}(\hat{v}_i)} \lambda_{max}\left(\sum_i v_i v_i^T\right)$$

$$\theta(\hat{v}_1, \ldots, \hat{v}_n) \leq (1+\sqrt{\alpha})^2$$

<u>Ideally:</u> If all resulting characteristic polynomial had common interlacer, we could give this guarantee. $\Rightarrow$ $\{f_i\}$ all possible characteristic polynomials

- common interlacer $\Rightarrow$ all cvx. combination real roots
- $\mathbb{E}\, P$ is cvx. combination & has real roots.

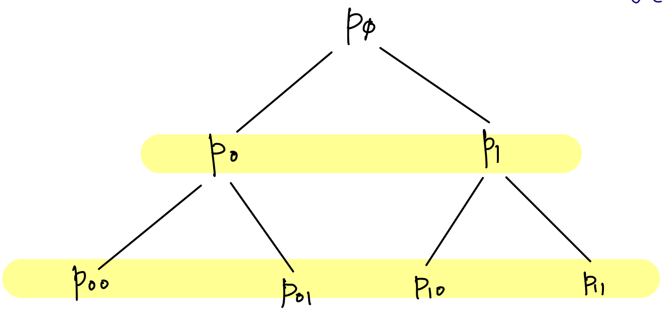
Too much to hope for

* <u>Thm</u>: Suppose $f_1, \ldots, f_m$ are real-rooted of degree n with positive leading coef. $\lambda_k(f_j)$: k-th largest root of f_j
μ be any dist on $[m]$.
If $f_1, \ldots, f_m$ have common interlacing
then $\forall k = 1, \ldots, n$

$$\min_j \lambda_k(f_j) \leq \lambda_k\left(\mathbb{E}_{j \sim \mu} f_j\right) \leq \max_j \lambda_k(f_j).$$

INTERLACING FAMILY

$$\theta(\hat{v}_1, \ldots, \hat{v}_n) = \min_{v_i \in supp(\hat{v}_i)} \lambda_{max}\left(\sum v_i v_i^T\right)$$



- Def: Interlacing family:

  Connected tree, where each node is the common interlacer with its childeren

$\Rightarrow$ Every interlacing family contains leaf nodes p_{leaf_1}, p_{leaf_2} s.t.

$$\lambda_k(p_{leaf_1}) \le \lambda_k(p_{top}) \le \lambda_k(p_{leaf_2})$$

PUTTING IT TOGETHER

If there exists interlacing family with

$$\left\{ \chi_{\sum v_i v_i^T}(x) \right\}_{v_i \in \mathrm{supp}(\hat{v}_i)} \quad \text{as leaf nodes}$$

AND

$$\mathbb{E}\left\{ \chi_{\sum_i v_i v_i^T}(x) \right\} \quad \text{as top node}$$

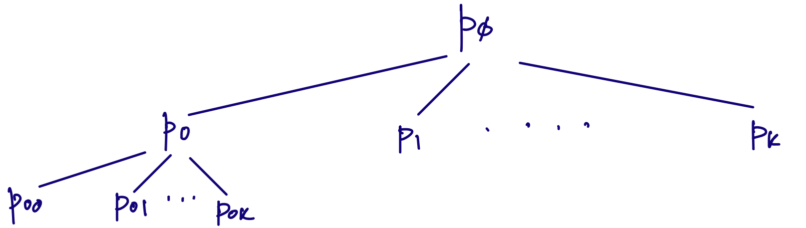
Then

$$\theta(\hat{v}_1, \ldots, \hat{v}_n) \leq \mathrm{maxroot}\left\{ \mathbb{E}\left\{ \chi_{\sum_i v_i v_i^T}(x) \right\} \right\}.$$

BUILDING SUCH TREES

Let $\hat{V} = \sum \hat{v}_i \hat{v}_i^T$

Going down tree: Revealing value of each $\hat{v}_i$



$\sum v_i v_i^T$

where

$$p_{s_1 s_2 \dots s_r} = \mathbb{E}\left\{ X_{\hat{V}} \;\middle|\; \hat{v}_1 = s_1, \dots, \hat{v}_r = s_r \right\}$$

$\Rightarrow$ Siblings at depth t, differ at $\hat{v}_t$.

* <u>Theorem</u>: Let $\hat{v}_1, \dots, \hat{v}_m$ be indep rand vec s.t.

$$\mathbb{E}[\hat{v}_i \hat{v}_i^T] = A_i$$

. Then

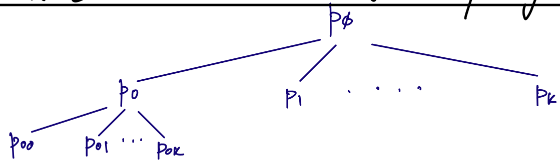
$$\mathbb{E}\{\chi_{\hat{v}}(x)\} = \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) \det\left[xI + \sum_{i=1}^{m} z_i A_i\right] \Bigg|_{z_1 = \cdots = z_m = 0}$$

→ Expectation depends only on expected outer prod of random vec.

Call this mixed characteristic polynomial denoted by

$$\mu[A_1, \dots, A_n](x)$$

- Claim: Every polynomial we saw in a tree is a mixed characteristic polynomial.



where $p_{s_1 s_2 \dots s_r} = \mathbb{E}\{\chi_{\hat{v}} \mid \hat{v}_1 = s_1, \dots, \hat{v}_r = s_r\}$

- Leaf polynomials: $(\sigma_i = v_i \text{ for } i \in [m])$

$$p_\sigma(x) = \chi_{\sum v_i v_i^T}(x) = \mu[v_1 v_1^T, \dots, v_m v_m^T](x)$$

- Top polynomials: $\mathbb{E}[\chi_{\hat{v}}(x)] = \mu[A_1, \dots, A_m](x)$

- Middle polynomials: $(\sigma_i' = v_i \text{ for } i \in [k])$

$$p_{\sigma'} = \mathbb{E}\{\chi_{\hat{v}}(x) \mid \hat{v}_i = \sigma_i, i \in [k]\}$$

TBC.