Counting Bases of a Matroid.

(Anari, Liu, Oveis Gharan, Vinzant)

STOC 2019

Apoorv Vikram Singh

Sept 16, 2022

Markov Chains

- Sequence of r.v. $X_t \equiv$ M.C. if

$$\mathbb{P}(X_{t+1}=y \mid X_0=x_0, \dots, X_t=x_t) = \mathbb{P}(X_{t+1}=y \mid X_t=x_t)$$

- $$P(x,y) = \mathbb{P}(X_{t+1}=y \mid X_t=x)$$

$$x \longrightarrow y$$

- t-step distribution:

$$P^t(x,y) = \begin{cases} P(x,y), & t=1 \\ \sum_{z\in\Omega} P(x,z)\,P^{t-1}(z,y), & t>1 \end{cases}$$

- Stationary Distribution:

$$\forall y \in \Omega, \quad \pi(y) = \sum_{x \in \Omega} \pi(x) P(x,y)$$

- Ergodic M.C.:

$$\exists t \quad \text{s.t.} \quad \forall x, y \in \Omega, \quad P^t(x,y) > 0.$$

* <u>Theorem</u>: For a finite, ergodic M.C. there exists a unique stationary dist. π s.t.

$$\forall x, y \in \Omega, \quad \lim_{t \to \infty} P^t(x,y) = \pi(y).$$

* **Rmk**: For ergodic MC when P is symmetric

$\forall x, y \in \Omega \qquad P(x,y) = P(y,x)$

then π is uniform over Ω.

* **Mixing Time**:

$$T_{mix}(\varepsilon) := \max_{X_0 \in \Omega} \min \left\{ t : d_{TV}\left(P^t(X_0, \cdot), \pi\right) \leq \varepsilon \right\}$$

$\Rightarrow$ Time until chain "mixes" i.e. within TV dist $\leq \varepsilon$ from worst initial state.

- REMARKS:

1) Showed how one can get uniform Stationary dist. over Ω

2) Idea: Run M.C. for time T_{mix} and then sample from it.
   = almost unif sampling.

- Approximate counting & Almost unif Sampling are related.

$$\begin{array}{ccc} \text{Exact Counter} & \Longrightarrow & \text{Exact Sampler} \\ \Downarrow & & \Downarrow \\ \text{Approx Counter} & \Longleftrightarrow & \text{Approx. Sampler} \end{array}$$

E.G.

* Counting Matchings in a graph $|\mathcal{M}(G)|$

- $G=(V,E)$ $\quad e_1,\dots,e_m\in E$.

- Consider seq. of graphs

$$G_0=G,\quad G_1=G_0\setminus e_1,\ \dots,\ G_i=G_{i-1}\setminus e_i,\ \dots$$

$$G_m=G_{m-1}\setminus e_m=(V,\phi)$$

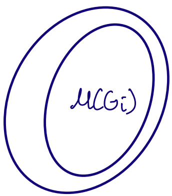
α $|\mathcal{M}(G_m)|=1$.

$|\mathcal{M}(G_0)|=|\mathcal{M}(G)|$.

$$\frac{1}{|\mathcal{M}(G)|}=\frac{|\mathcal{M}(G_1)|}{|\mathcal{M}(G_0)|}\cdot\frac{|\mathcal{M}(G_2)|}{|\mathcal{M}(G_1)|}\cdot\dots\cdot\frac{|\mathcal{M}(G_m)|}{|\mathcal{M}(G_{m-1})|}$$

$$p_i=\frac{|\mathcal{M}(G_i)|}{|\mathcal{M}(G_{i-1})|}\qquad\therefore\quad |\mathcal{M}(G)|=\prod_i 1/p_i\,.$$

• Bounding p_i : 1) $\mathcal{M}(G_i) \subseteq \mathcal{M}(G_{i-1})$



$\mathcal{M}(G_{i-1})$

2) $\forall M \in \mathcal{M}(G_{i-1}) \setminus \mathcal{M}(G_i)$

$$\Downarrow$$

$$M \setminus \{e_{i-1}\} \in \mathcal{M}(G_i)$$

$\therefore$

$$\frac{1}{2} \leq p_i \leq 1$$

$\therefore$ Aim: Find additive approx to p_i to within error $\left(1 \pm \frac{\varepsilon}{4m}\right)$ multiplicatively w.p. $\left(1 - \frac{\delta}{m}\right)$

$\therefore$ $\Rightarrow$ $\left(1 \pm \frac{\varepsilon}{2m}\right)$ error to $\frac{1}{p_i}$

$\therefore$ Taking product we obtain

$\left(1 + \frac{\varepsilon}{2m}\right)^m < 1 + \varepsilon$ approx to $|\mathcal{M}(G)|$ w.p. $1 - \delta$.

$\because$ $\frac{1}{2} \leq p_i \leq 1$ $\Rightarrow$ estimate p_i to additive error $\pm \frac{\varepsilon}{8m}$

$\therefore$ Need to approx p_i.

APPROXIMATING p_i

- The fraction of matchings in G_{i-1} that do NOT contain $\{e_i\}$ to error $\pm \varepsilon/8m$.

- Idea: Use Approx Sampler. $\pi = \text{unif}(\mathcal{M}(G_{i-1}))$

$$A := \{ M \in G_{i-1} : e_i \notin M \}$$

- Use approx unif sampler : μ s.t. $\|\mu - \pi\|_{TV} \le \varepsilon/10m$

$$\left| \mathbb{P}_{M \sim \mu}[M \in A] - \underbrace{\mathbb{P}_{M \sim \pi}[M \in A]}_{p_i} \right| \le \frac{\varepsilon}{20m}$$

$$\Rightarrow \quad p_i - \frac{\varepsilon}{20m} \le \mathbb{P}_{M \sim \mu}[M \in A] \le p_i + \frac{\varepsilon}{20m}$$

$\therefore$ By Chernoff, it is enough to generate

$O\left(\left(\frac{m}{\varepsilon^2}\right)\log\left(\frac{m}{\delta}\right)\right)$ matchings from $\mathcal{U}$

and compute fractions that are in A.

(To find $\frac{\varepsilon}{20m}$ additive approx to $\mathbb{P}_{M\sim\mathcal{U}}[M\in A]$)

$\Downarrow$

($\frac{\varepsilon}{10m}$ additive approx to p_i)

Total complexity (samples):

$$\text{poly}\log\left(\frac{cm^2}{\varepsilon^2}\right).$$

STATUS:

Approx Unif Sampling

$\Downarrow$

Approx Counting

Better than birthday paradox argument

⊠ Counting Bases of a Matroid

* <u>Matroid</u>

[I] Examples:

(G, I)

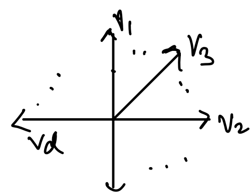
- $G = (V, E)$

$I = \{ S \subseteq E \mid S \text{ contains no cycles} \}$

Maximum size S are called Spanning Trees.

- <u>Linear Matroid</u>

$I = \{ S \subseteq \{v_1, \ldots, v_n\} \mid S \text{ has linear indep.} \}$



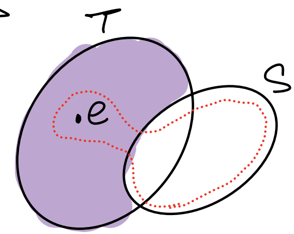
$\boxed{\text{II}}$ Definition: MATROID

- $[n] = \{1, \dots, n\} \to$ ground set

$I \subseteq 2^{[n]}$ s.t.

Prop 1: I is downward closed / simplicial cx.

$$S \in I,\ T \subseteq S \Rightarrow T \in I$$



Prop 2: (Exchange)

$$\left.\begin{array}{l} S, T \in I \\ |T| > |S| \end{array}\right\} \Rightarrow \exists\, e \in T \setminus S \text{ s.t. } S + e \in I.$$

- Def (Basis): Maximal $S \in \mathcal{I}$ } exchange prop.

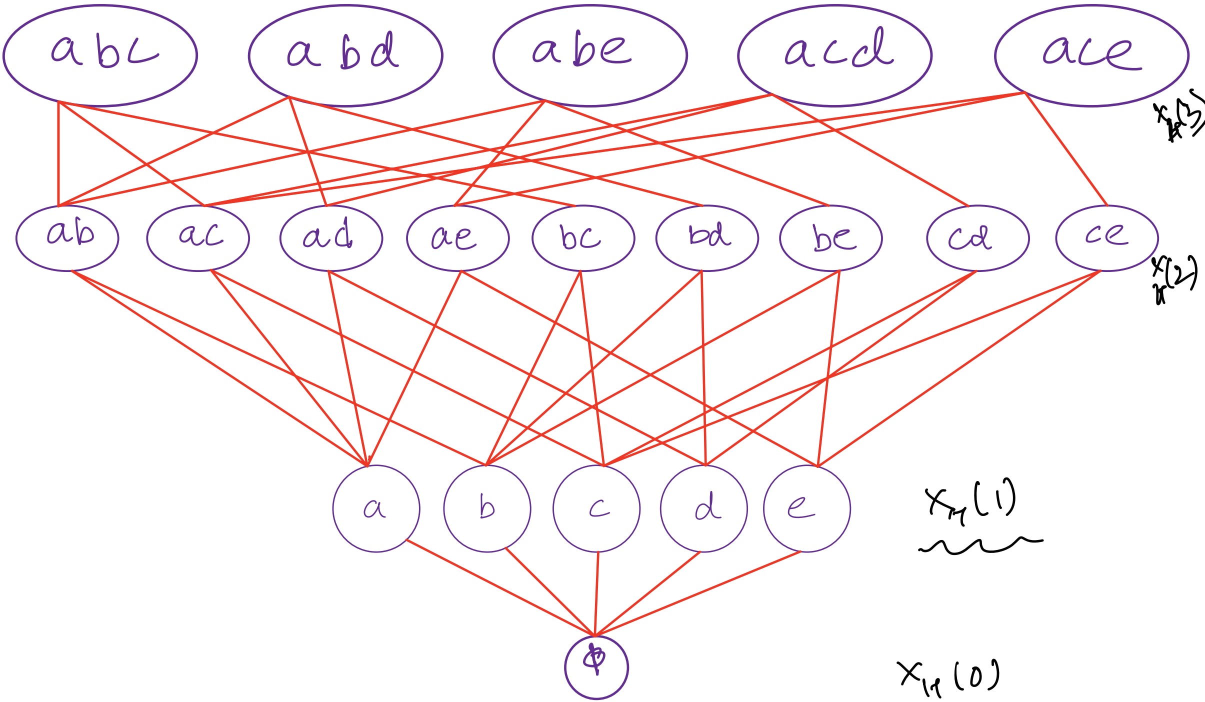
  Rank = common size of bases.

- Prop'l: Bases uniquely determine a matroid. }

Downward close

- **Picture:** $[n] = \{a, b, c, d, e\}$

  $B = \{abc, abd, abe, acd, ace\}$



Geometric Definition of a Matroid

<u>AIM</u>: <u>Given</u>: Collection of sets F from ground set $[n]$.

<u>Determine</u>: If they form basis of some matroid

<u>APPROACH</u>: Consider the set of indicator vectors of Basis.

$$\left\{ \mathbb{1}_B \,:\, B \in F \right\} \qquad \mathbb{1}_B = (0, 0, 1, 0, 1, \dots, 0, 1, 0)$$

items in basis

- Consider the Matroid polytope

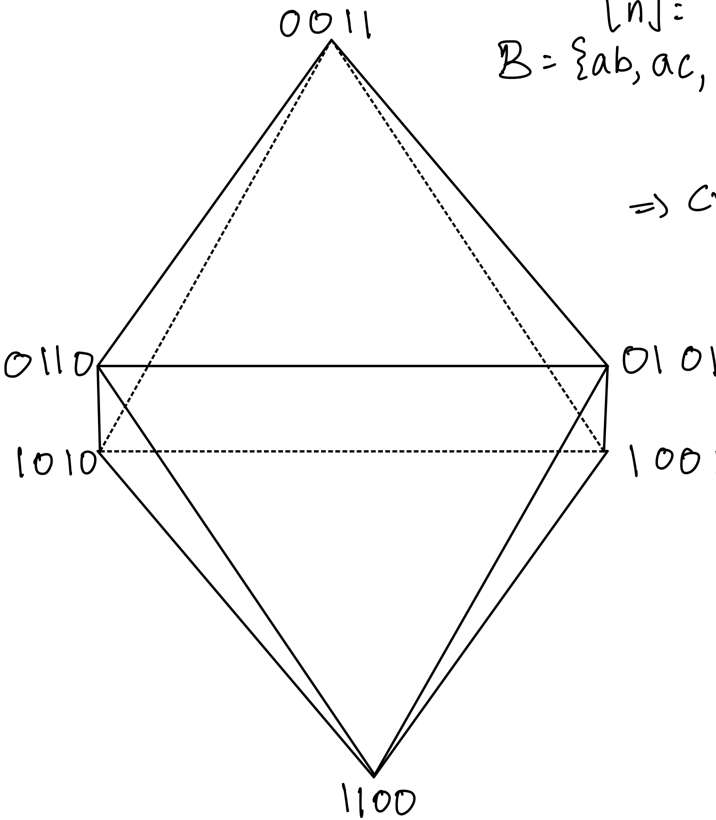
$$\text{conv-hull}\left(\left\{ \mathbb{1}_B : B \in F \right\} \right)$$

PICTURE:

$[n]: \{a, b, c, d\}$

$\mathcal{B} = \{ab, ac, ad, bc, bd, cd\}$

$\Rightarrow \operatorname{cvx}(\mathbb{1}_B \mid B \in F)$



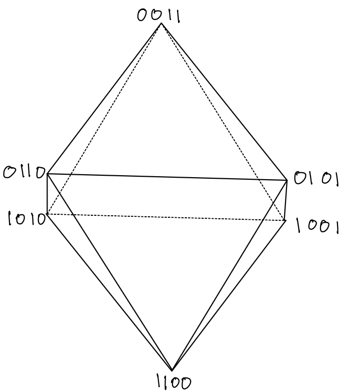
* **Theorem**:

F is a collection of bases of some matroid

$$\Updownarrow$$

All edges (1-dimn faces) are parallel to

$e_i - e_j$, $i, j \in [n]$



Proof: • A & B neighbours $\exists$ linear func'n w maximized on edge b/w A & B

• Exchange thm says: $a \in A$, $b \in B$ s.t. $(A \setminus a) \cup b$ & $(B \setminus b) \cup A$ are Bases

• Linearity: $w(A) + w(B) = w(A \setminus a \cup b) + w(B \setminus b \cup a)$

$\Rightarrow$ $A \setminus a \cup b = B$ $\because$ w only max. on that edge.

* Corollary: Dual of a matroid is a matroid

$$\{ [n] - B \mid B \text{ basis} \}$$

Pf: · Transform 0's to 1's in polytope ⇒ Linear transform.
· ∴ Edges of polytope don't change under lin transf.

• Corollary: Bases restricted to face of polytope also form basis of some matroid.

Recall GOAL:	Sample basis of matroid randomly (or) approximately count the number of basis

* **Theorem**: Down-up walk on bases of matroid has $\underbrace{\text{spectral gap}}_{1-\lambda_2(\text{r.w.})} \geq \dfrac{1}{(\text{rank})}$

Down-Up walk: (Not typical r.w. on polytope)

$$S \xrightarrow[\text{u.a.r}]{e \text{ chosen}} S \setminus e \xrightarrow[\text{those that give a basis}]{f \text{ chosen u.a.r. from}} S \setminus e + f$$

$\left(\text{Mixing time} \approx \dfrac{1}{1-\lambda_2(\text{r.w.})} \log(\#\text{ states}) \right)$

* **Corollary**: Mixing time $= O(\text{rank} \cdot \log(\#\text{ basis}))$

$\leq O(\text{rank}^2 \log n)$

(Guo, Cryan, Mousa '20): Mixing time $= O(\text{rank} \log \text{rank})$

* ## Proof Sketch :

1. Links of simplicial complex are also matroids
2. Top links are 0-local expanders
3. Profit based on trickle-down & local-to-global.

Crash Course on High Dimensional Expanders

- **Def** (d-unif hypergraph): $H \stackrel{=}{} (V,E)$: on vertex set $V = [n]$

$$E \subseteq \binom{V}{d}$$

- **Def** (simplicial cplx. associated with H)

$$X_H = X_H(0) \cup \cdots \cup X_H(d) \rightarrow$$ Given by downward closure of H

- $X_H(d) = H \subseteq \binom{[n]}{d}$ • $X_H(i)$ consists $\tau \in \binom{[n]}{i}$

s.t. $\tau \subset T$ for $T \in X(d)$.

Q: When is pure simplicial cplx a High Dimn. Expander?
(HDX)

- Technique called local-to-global paradigm
  - When HDX satisfies property locally

    $\Downarrow$

    Satisfies (weaker version) of that property globally.

- LINKS: Let $\tau \in X(k)$, $0 \le k \le d$.
  - Link of τ: $\rightarrow$ local view of sets of node of a graph.
  - If all local links expand $\Rightarrow$ HDX

• **Def (Links):** $\tau \in X(k)$

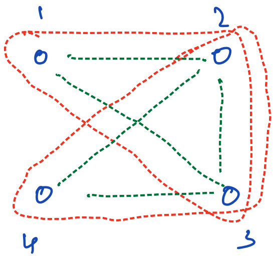
Link of τ = $X_\tau := \{\sigma : \sigma \cup \tau \in X, \ \sigma \cap \tau = \phi\}$

- ⇒ Set of stuff (σ) you add in τ s.t. it is in X but disjoint from τ.

- X_τ is $(d-k)$-dimn simplicial complex

PICTURE:

eg: 3 dimn Hypergraph



$X = X(3) \cup X(2) \cup X(1)$

green : 2 dimn $X(2)$
red : 3 dimn $X(3)$
blue : 1 dimn $X(1)$

$X(3) : \{123, 234\}$
$X(2) : \{12, 23, 13, 23, 34, 24\}$
$X(1) : \{1, 2, 3, 4\}$

Link of $\{1\}$ ⇒ link is $3-1 = 2$ dimn ∴ $X_1(2)$.

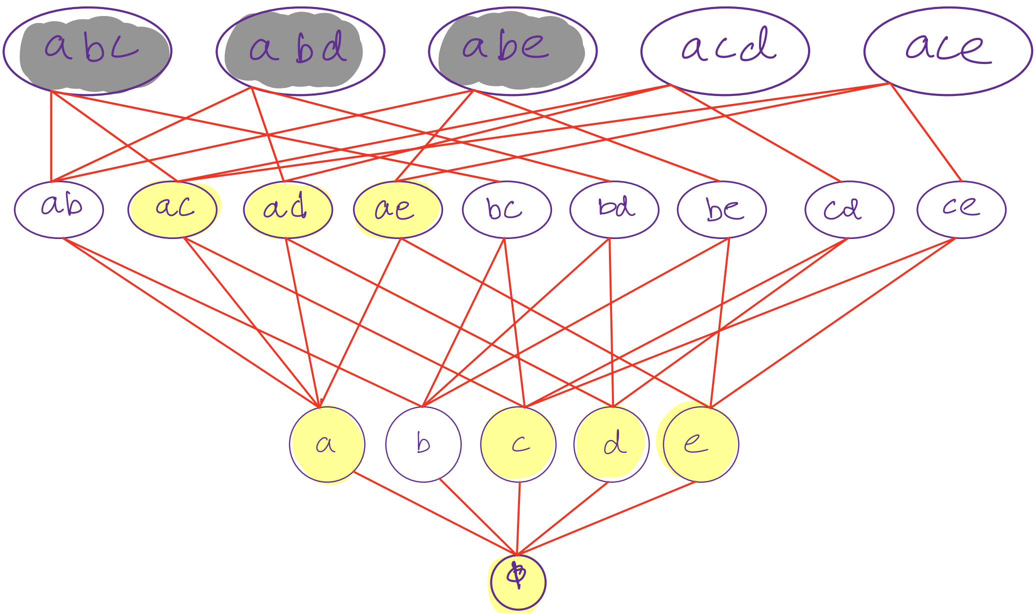
$X_1(2) = \{23\}$
$X_1(1) = \{2, 3\}$

Link of $\{12\}$ ⇒ link is $3-2 = 1$ dimn ∴ $X_{12}(1)$

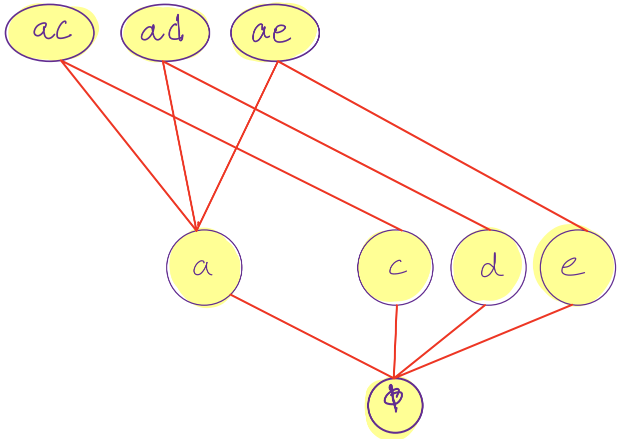
$X_{12}(1) = \{3\}$

Q: What is the link of a node of graph?

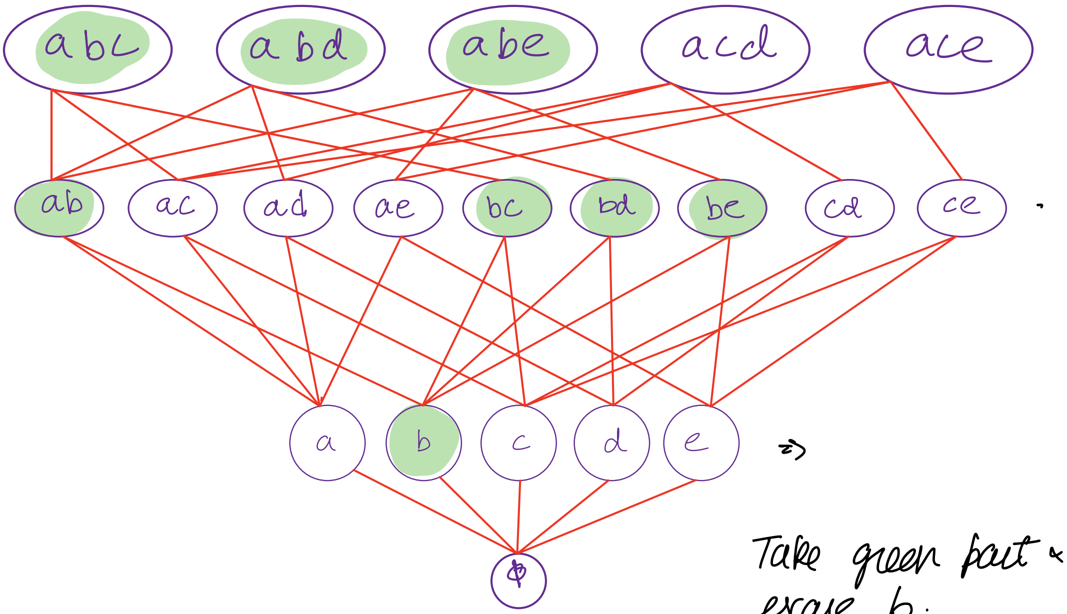
Let us see some examples of a link: Link of b.



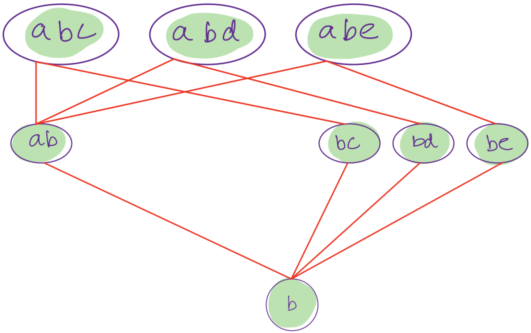
$\therefore$ Link of b is Ξ



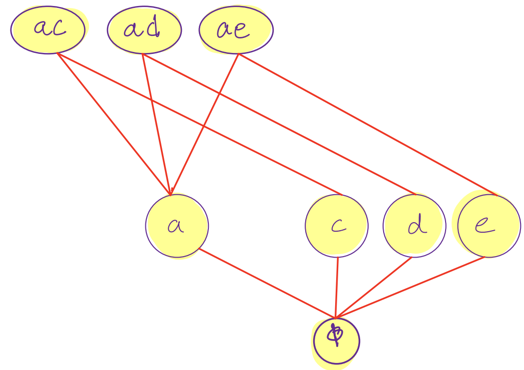
Link of b : another view



$\Rightarrow$ Take green part & erase b.

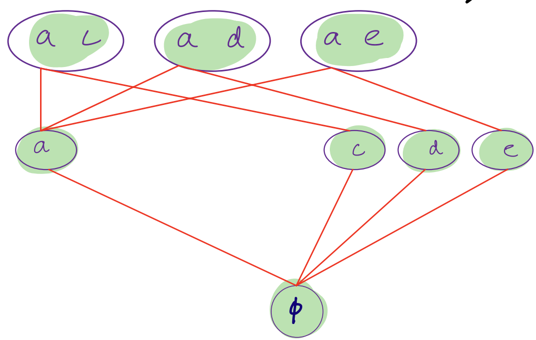


$\equiv$

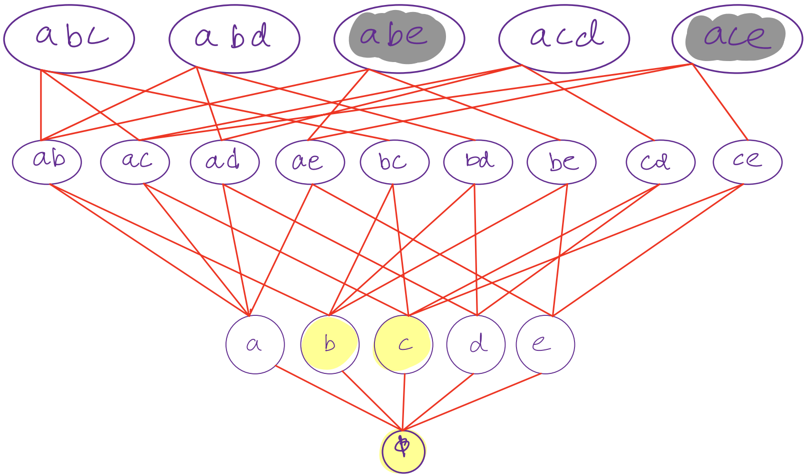


erasing b

$X_b \equiv$

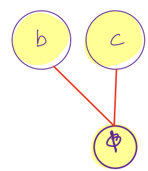


* Link Example: Link of ae



$\equiv X$

$X_{ae} \equiv$



* <u>Why did we bother with links so much?</u>

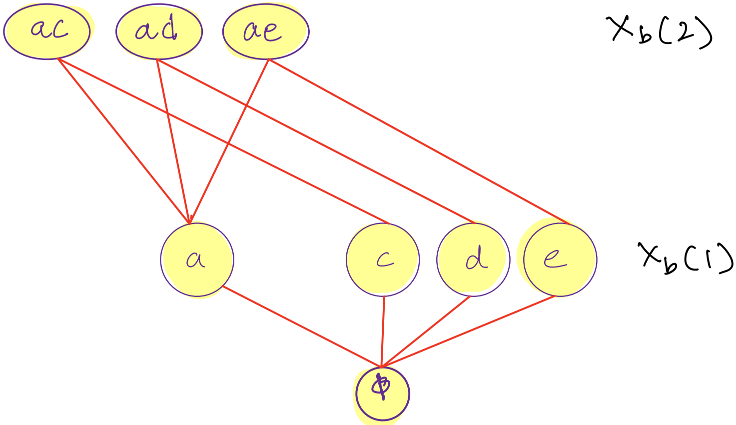
* Informal Statement:

  · If all local links expand $\Rightarrow$ HDX

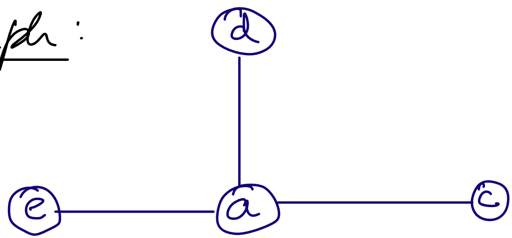
* What does it mean for a link to expand?

  - Link is a simplicial complex.
  - Just look at vertices & edges of this complex and see if they expand; i.e. $X_\tau(1) \cup X_\tau(2)$.

Recall $X_b \equiv$



$\Rightarrow$ Corresponding Graph :



$\rightarrow$ Check if this expands or not.

* <u>Definitions</u>:

$(1 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq -1)$

1. <u>Weighted Spectral Expanders</u>: $G=(V,E)$ is

λ-spectral expander if:

$$\lambda_2 \leq \lambda$$

2. <u>γ- local Spectral Expander</u>:

A weighted simplicial cplx. is γ-local spec. expa. if underlying graph of every i-link for

$0 \leq i \leq d-2$ is γ-spectral expander.

* ## Oppenheim's Trickling Down Theorem

IDEA: Intuitively $\because$ we have downward closedness top links should be enough to tell if its expanding.

* **Thm**: Let (X, Π) be a d-dimn simplicial cplx:

1. Every $(d-2)$-link is γ- spectral expander.

2. $\forall$ i-links are connected for $0 \leq i \leq d-2$.

Then, (X, Π) is a

$$\frac{\gamma}{1-(d-2)\gamma} \text{ - local spectral expander.}$$

$\therefore$ Owing to Trickling down theorem - it suffices to ensure :

- Top links are expanding.
- Complex is connected.

$\Bigg\} \Rightarrow$ cplx. is expanding.

The Picture :

1. We want to analyse mixing time of Down-up walk.

$$S \xrightarrow{e \text{ chosen u.a.r}} S \setminus e \xrightarrow{f \text{ chosen u.a.r. from those that give a basis}} S \setminus e + f$$

i.e. λ_2 of this r.w. matrix

$\because$ (FOLKLORE)

$$\text{Mixing time} \approx \frac{1}{1 - \lambda_2(\text{r.w.})} \log(\#\text{ states}) .$$

2. Fortunately in 2018, Kaufman & Oppenheim proved spectral gap results for

Down-up walks in general.

3. ALOV '19 : Used KO'18.

* Thm [Kaufman & Oppenheim 2018]:

Let (X, Π) be a pure d-dimn 0-local spectral expander. and let $0 \le k < d$.

Then, second-largest eig. of down-up walk on a k-face is at most

$$\lambda_2(P_k^{\vee}) \le 1 - \frac{1}{k}.$$

* Now we have all the tools to analyze matroid sampling algorithm.

- RECALL PROOF SKETCH:

$\boxed{A}$ Links of simplicial complex are also matroids.

$\boxed{B}$ Top links are 0-local expanders.

$\boxed{C}$ Profit based on trickle-down & local-to-global.

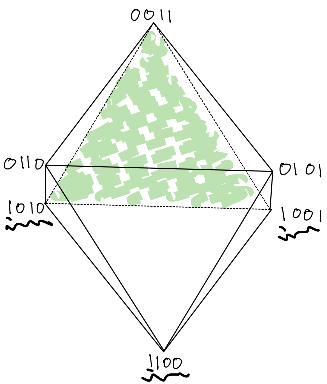
A Links of simplicial cplx are also matroids:

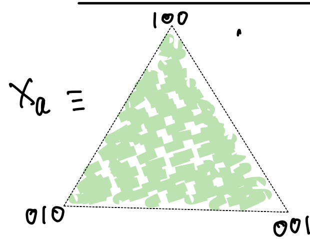
- Link X_τ of $\tau \in X(k)$ is

  $(d-k)$-dimn. simplicial cplx.

- In matroid language:

  Link corresponds to matroid restricted to a face, which is also a matroid.





(a, b, c, d)

This is a matroid ✓

Link is X_a : first coordinate.

B Top links are 0-local expanders

- Now we need to prove that matroids that are top links are 0-local spectral expanders.

→ Need this to use trickle down theorem

* **Thm**: Let (X, Π) be a d-dimn simplicial cplx:
  1. Every $(d-2)$-link is γ-spectral expander.
  2. $\forall$ i-links are connected for $0 \leq i \leq d-2$.

  Then, (X, Π) is a

$$\frac{\gamma}{1-(d-2)\gamma} \text{ - local spectral expander.}$$

$(d-2)$

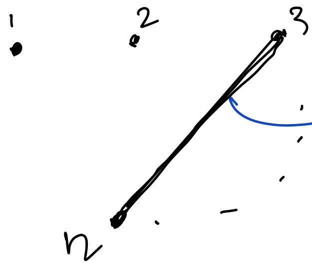
$d-(d-2)=2$

→ In matroid language, top-links are rank 2 matroids.

Good News: We know full characterization of rank 2 matroids.

T.P.: λ_2 (r.w. on rank 2 matroid) ≤ 0.

* **Characterization**:



edge signifies that $\{3, n\}$ is a basis

Thm: Graph is a matroid of rank 2

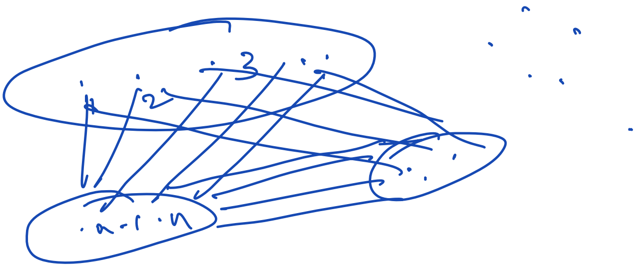
$$\Updownarrow$$

complete multipartite graph + isolated vertices.

Thm: Graph is a matroid of rank 2

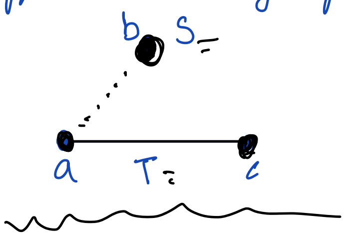
$\Updownarrow$

complete multipartite graph + isolated vertices.



$\Rightarrow$ Follows from exchange property: $\exists\, e \in T \setminus S$ s.t. $S \cup e \in \mathcal{I}$

eg.



$\Rightarrow$ Forbidden config.

$T \setminus S = \{a, c\}$

$\Rightarrow$ $\therefore$ Consider equivalence relationship $v \sim u$

$v \sim u$: if $\{u, v\}$ not an edge.

Then $a \sim b$, $b \sim c$ $\Rightarrow$ $a \sim c$

$\therefore$ Now we look at, λ_2(adj of complete multipartite graph) ≤ 0

$$A = \begin{bmatrix} 0 & & 1 \\ & \boxed{0} & \\ 1 & & \boxed{0} \end{bmatrix} = \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} - \begin{bmatrix} 1 & & 0 \\ & \boxed{1} & \\ 0 & & \boxed{1} \end{bmatrix}$$

$\therefore$ $\lambda_2(A) \leq 0$

But we need to show λ_2 for r.w. matrix, not adjacency matrix.

* <u>Lemma</u>: Let symm $M \geq 0$, TFAE:

- $\lambda_2(M) \leq 0$
- $\exists$ $(n-1)$ dimn. sp. V s.t. $\forall x \in V$: $x^T M x \leq 0$.
- Let $v \in \mathbb{R}^n_{>0}$ let $V = \{x \mid x^T M v = 0\}$

  $\therefore$ $\forall x \in V$, $x^T M x \leq 0$
- $\forall v \in \mathbb{R}^n_{>0}$: $(x^T M x)(v^T M v) \leq (x^T M v)^2$

⇒ Using this we can prove that

If $\lambda_2(M) \leq 0$, $M \geq 0$, D diag with ≥ 0

then $\lambda_2(DMD) \leq 0$

$$\Downarrow$$

$$\lambda_2(D^{-1/2} M D^{-1/2}) \leq 0$$

and

$$\underbrace{D^{-1}M}_{\hookrightarrow \text{ r.w. matrix}} \ \underset{\sim}{\text{similar}} \ D^{-1/2} M D^{-1/2}$$

$$\therefore \ \lambda_2(D^{-1}M) \leq 0 \qquad \blacksquare$$

* Putting things together

1. Links of simplicial cplx. are matroids.
2. Top links are 0-local spectral expanders
   (by showing rank 2 matroid connection)
3. Everything is connected in our example.
4. Use Trickle-down thm. of Oppenheim to imply simplicial cplx. are 0-local spectral exp.
5. Use Kaufman & Oppenheim to give bound on λ_2 (down up walk).

$\Rightarrow$ Mixing time $\Rightarrow$ Unif sampling $\Rightarrow$ Counting