3.22 ③

① $\|A\|_F^2 = \sigma_1^2 + \cdots + \sigma_k^2 + \sigma_{k+1}^2 + \cdots + \sigma_n^2$

$\Rightarrow \quad \sigma_k^2 + \cdots + \sigma_k^2 + \sigma_{k+1}^2 + \cdots + \sigma_n^2 \le \|A\|_F^2 \qquad \because (\sigma_i \geq \sigma_j \text{ for } i<j)$

$\|A\|_F^2 \geq k\sigma_k^2 + \sigma_{k+1}^2 + \cdots + \sigma_n^2$

$\Rightarrow \|A\|_F^2 \geq k\sigma_k^2$

$\therefore \quad \sigma_k \le \dfrac{\|A\|_F}{\sqrt{k}}.$

② $\|A-B\|_2 \le \sigma_k \le \dfrac{\|A\|_F}{\sqrt{k}}$

~~Take~~ Let SVD decomposition of $A$ be $U\Sigma V^T$.

i.e. $\sum_{i=1}^{n} \sigma_i u_i v_i^T$

Take $B = \sum_{i=1}^{k} \sigma_i u_i v_i^T \Rightarrow B$ is a rank $k$-matrix.

$\therefore \quad A - B = \sum_{i=1}^{n} \sigma_i u_i v_i^T - \sum_{i=1}^{k} \sigma_i u_i v_i^T$

$= \sum_{i=k+1}^{n} \sigma_i u_i v_i^T$

$\therefore \|A-B\|_2 = \sigma_{k+1} \le \sigma_k \le \dfrac{\|A\|_F}{\sqrt{k}}.$

③ Can we replace 2 by Frob. norm? i.e. is $\|A-B\|_F \le \dfrac{\|A\|_F}{\sqrt{k}}$ for a rank $k$, $B$ matrix

~~$\|A-B\|_F^2 = \sigma_{k+1}^2 + \cdots + \sigma_n^2$~~ $\|A-B\|_F^2 = \sigma_{k+1}^2 + \cdots + \sigma_n^2$

compare it with $\dfrac{\sigma_1^2 + \cdots + \sigma_n^2}{k}$. Take eg. $k=2$ & $n = 12$, and $\sigma_1 = \cdots = \sigma_n = 1$.

$\|A-B\|_F^2 = 10$

$\|A\|_F^2 = 12$

$k = 2$

$\therefore \|A-B\|_F^2 = 10$ v/s $\dfrac{\|A\|_F^2}{2} = 6$

$10 > 6$ $\therefore$ cannot replace 2 by a Frob. norm.

Think: why can I take B to be just $\sum_{i=1}^{k} \sigma_i u_i v_i^T$ ?