Q2) $f: \mathbb{R} \to \mathbb{R}$, β-smooth, α-strg. cvx.

$\Rightarrow$ For ~~β~~ smoothness $\Rightarrow$ Look at $\|\nabla g(x) - \nabla g(y)\| \leq G_1 \cdot \|x-y\|$
What is the value of G_1?

We know that

$g(x) = f(cx)$. Since $f(cx)$ is β-smooth & α-str. cvx,
this means that f is differentiable.

$\therefore \quad g'(x) = c f'(cx)$

$$
\begin{aligned}
|g'(x) - g'(y)| &= |c f'(cx) - c f'(cy)| \\
&= c\,|f'(cx) - f'(cy)| \\
&\leq c \cdot \beta\,|cx - cy| \\
&= c^2 \beta\,|x-y|
\end{aligned}
$$

$\therefore$ g is $c^2\beta$ smooth.

$\Rightarrow$ Strong convexity.

$g(y) - g(x) - g'(x)(y-x) \geq \frac{G_2}{2} \|x-y\|^2$

$= f(cy) - f(cx) - c f'(cx)(y-x) \quad - (1)$

We know that (from strong cvx of f)

$f(cy) - f(cx) - \cancel{c} f'(cx)(cy - cx) \geq \frac{\alpha}{2} c^2 (x-y)^2$

$\Rightarrow f(cy) - f(cx) - c f'(cx)(y-x) \geq \frac{\alpha c^2}{2} (x-y)^2 \quad - (2)$

$\therefore$ (1) & LHS of (2) are the same

$\therefore$ g is αc^2 strongly convex.

$\therefore$ condition number of g is

$$\frac{c^2\beta}{c^2\alpha} = \beta/\alpha \Rightarrow \text{condition number of } f.$$