~~eg: $Y = \boxed{6}$~~

6) $X \in \mathbb{R}^{n \times d}$. $\quad X = U \Sigma V^T$, $\sigma_1, \dots, \sigma_d$

$$\underbrace{(X^TX) \cdot (X^TX) \cdots (X^TX)}_{q\text{-times}} = \cancel{(U\Sigma V^T)(U\Sigma V^T) \cdots (U\Sigma V^T)}$$

$$X^TX = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T \qquad (\because U^TU = I)$$

$$\therefore (X^TX)^q = \underbrace{(V\Sigma^2V^T)(V\Sigma^2V^T) \cdots (V\Sigma^2V^T)}_{q\text{-times}} \qquad \because (V^TV = I)$$

$$= V \Sigma^{2q} V^T.$$

Ill^ly

$$XX^T = U \Sigma V^T V \Sigma U^T = U \Sigma^2 U^T$$

$$(XX^T)^q = \underbrace{(U\Sigma^2U^T)(U\Sigma^2U^T) \cdots (U\Sigma^2U^T)}_{q\text{-times}}$$

$$= U \Sigma^{2q} U^T.$$

$\therefore$ Eigenvalues of $(X^TX)^q$ are $\sigma_1(X)^{2q}, \dots, \sigma_d(X)^{2q}$

Eigenvectors of $(X^TX)^q$ are column vectors of V.

Eigenvalues of $(XX^T)^q$ are $\sigma_1(X)^{2q}, \dots, \sigma_d(X)^{2q}$

Eigenvectors of $(XX^T)^q$ are column vectors of U.

- Time required to compute $(X^TX)^q y$ for some vector y

  $X \in \mathbb{R}^{n\times d}$

$$(X^TX)^q y = X^TX\,X^TX\left(X^T\left(X\cdots\left(X^TXy\right)\right)\right)$$

$\rightarrow$ cost to compute $Xy = O(nd)$

cost to compute $X^T(Xy) \rightarrow$ first compute (Xy), then $X^T(Xy)$ And so on

$\Rightarrow O(nd) + O(nd)$

$\therefore$ Total time to compute $~~(XX^T)y~~$ $(X^TX)^q y$ or $(XX^T)^q z$ is $O(ndq)$.